that, in our theory, we use one more expansion term, i.e., $\delta^2 g_2(\eta)$ in Eq. (4), which corresponds to an amount of $-(\gamma+1)\delta^2/32\xi_0$ (γ is the ratio of the specific heats) in the shock position on the axis for two-dimensional case. The second reason is that the one-dimensional shock shape $x_s = x_0$ in Ref. 4 is a straight line parallel to the y axis, whereas in our theory the one-dimensional shock shape $\xi_s = \xi_0$ is a curve determined by Eq. (1). This produces an amount of $-\sinh\delta\xi_0 \cos\delta/\sin\delta \sim \frac{1}{2}\xi_0\delta^2$ in the shock position on the axis. The latter term is also neglected in Ref. 4 which keeps terms only up to $\mathcal{O}(\delta^{3/2})$. It is noted that the tangent of the shock at the nozzle wall is perpendicular to the x axis in Ref. 4, but is properly perpendicular to the nozzle wall in our theory. It can be deduced that the total error for Ref. 4 in the shock position on the axis is $\frac{1}{2}\delta^2[\xi_0(\gamma+1)/16\xi_0]$, which is about 1% for $R=10$ and $\xi_0=0.5$ and about 4% for $R=10$ and $\xi_0=1.0$.

Acknowledgment

The authors express thanks to Prof. D. Caughey for discussion and his efforts in providing the finite difference results used in this paper and also to a reviewer of the earlier version who pointed out our oversight of Refs. 4-7.

References

[1]Lin, C.Q. and Shen, S.F., "Two-Dimensional Transonic Nozzle Flows with Shock," *AIAA Journal*, Vol. 19, Nov. 1981, pp. 1494-1496.

[2]Oswatitsch, K. and Rothstein, W., "Flow Pattern in a Converging-Diverging Nozzle," NACA TM 1215, 1949.

[3]Hall, I.M., "Transonic Flow in Two-Dimensional and Axially-Symmetric Nozzles," *Quarterly Journal of Mechanics and Applied Mathematics,* Vol. XV, Pt. 4, 1962, pp. 487-508.

[4]Messiter, A.F. and Adamson, T.C., "On the Flow Near a Weak Shock Wave Downstream of a Nozzle Throat," *Journal of Fluid Mechanics*, Vol. 69, Pt. 1, 1975, pp. 97-108.

[5]Richey, G.K. and Adamson, T.C., "Analysis of Unsteady Transonic Channel Flow with Shock Waves," *AIAA Journal*, Vol. 14, Aug. 1976, pp. 1054-1061.

[6]Chan, J.S.K. and Adamson, T.C., "Unsteady Transonic Flows with Shock Waves in an Asymmetric Channel," *AIAA Journal*, Vol. 16, April 1978, pp. 377-384.

[7]Adamson, T.C., Messiter, A.F., and Liou, M.S., "Large Amplitude Shock-Wave Motion in Two-Dimensional, Transonic Channel Flows," *AIAA Journal*, Vol. 16, Dec. 1978, pp. 1240-1247.

[8]Lin, C.Q. and Shen, S.F., "Analysis of Transonic Flow with Shock in Slender Hyperbolic Nozzles," AIAA Paper 82-0160, 1982.

Material Contravariant Components: Vorticity Transport and Vortex Theorems

Luigi Morino*
Boston University, Boston, Massachusetts

Nomenclature

g_α = base vectors, Eq. (5)
J = Jacobian of transformation $x = x(\xi^\alpha)$
t = time
v = velocity
x = vector of Cartesian components x_i
z = see Eq. (8)
ξ^α = material curvilinear coordinates
ρ = density
ω^α = material contravariant components of ω
ω = vorticity

Subscripts
i = Cartesian components
α,β = covariant components

Superscripts
α,β = contravariant components

Received Dec. 10, 1984; revision received June 5, 1985.

*Professor of Aerospace and Mechanical Engineering and Director of Center for Computational and Applied Dynamics. Member AIAA.

THE objective of this Note is to introduce the concept of material contravariant components of vorticity (that is, the contravariant components of the vorticity in a convected curvilinear coordinate system), in order to show how classical vortex theorems may be interpreted in terms of these components and to indicate how these results may be used computationally. In particular, it is shown that, for inviscid isentropic flows, the material contravariant components of the vorticity divided by the density are constant following a material point. This result simplifies the proof and the kinematic interpretation of classical results, such as vortex stretching, Kelvin's theorem, as well as the fact that vortex lines are material lines. The result is then extended to general (viscous, nonisentropic) flows, and is used to obtain a simple but powerful computational scheme for the solution of Beltrami's vorticity equation.

Mathematical Preliminaries

In order to introduce the concept of material contravariant components, consider a material (or convected) coordinate system, ξ^α, i.e., a curvilinear coordinate system that is convected with the material particles. Note that Greek letters are used for curvilinear components (subscripts for covariant and superscripts for contravariant components), and subscript i for Cartesian components. Einstein summation convention on repeated indices is used on both Greek and Latin indices.

As in the classical formulation, a given material particle is identified by the material coordinates ξ^α and at any time its location may be determined by the Cartesian coordinates x_i, functions of the material coordinates, and time

$$x_i = x_i(\xi^1, \xi^2, \xi^3, t) = x_i(\xi^\alpha, t) \qquad (i=1,2,3) \tag{1}$$

Equation (1) gives the Lagrangian description of the fluid motion. It may be noted that if ξ^α ($\alpha=1,2,3$) are kept constant, then $x_i = x_i(t)$ represents the trajectory of a fluid point. As a consequence, coordinate lines and coordinate surfaces (for instance the surface $\xi^1 = \text{const}$) are always composed of the same particles and therefore are material lines and material surfaces, respectively.

It should be emphasized that the coordinates ξ^α are closely related to the classical material coordinate X^α (see e.g., Ref. 1, p. 128) which coincide with the Cartesian coordinates x_i at $t=0$. The only difference between the X^α and ξ^α coordinates is that, in contrast with the classical approach, it is not assumed herein that the ξ^α coordinates coincide with the Cartesian coordinates at time $t=0$. This not only emphasizes the curvilinear nature of the X^α and ξ^α coordinates, but, more importantly, yields an additional flexibility that allows for a convenient choice of the ξ^α coordinates at $t=0$, thereby facilitating the derivation of the abovementioned classical results.

Next, some elementary concepts on curvilinear coordinates are briefly reviewed and applied to the specific case of material curvilinear coordinates. The velocity of a material point is given by

$$v = \left.\frac{\partial x}{\partial t}\right|_{\xi^\alpha} \tag{2}$$

where $|_{\xi^\alpha}$ indicates that ξ^α ($\alpha = 1,2,3$) are kept constant while performing the differentiation. Next consider a quantity f, such as the density ρ, which is a function of space and time. The material (or substantial) derivative Df/Dt is the time derivative for an observer moving with the fluid particle, i.e.,

$$\frac{Df}{Dt} = \left.\frac{\partial f}{\partial t}\right|_{\xi^\alpha} = \left.\frac{\partial f}{\partial t}\right|_{x_i} + \frac{\partial f}{\partial x_i}\left.\frac{\partial x_i}{\partial t}\right|_{\xi^\alpha} \tag{3}$$

or, using Eq. (2),

$$\frac{Df}{Dt} = \left.\frac{\partial f}{\partial t}\right|_{x} + v\cdot \text{grad} f \tag{4}$$

This is true whether the quantity being differentiated is a scalar or a vector.

Next, for any given time t, introduce the base vectors

$$g_\alpha = \frac{\partial x}{\partial \xi^\alpha} \tag{5}$$

which are tangent to the coordinate lines. Recalling that the ξ^α lines are material lines, we see that the vectors g_α are always tangent to the same material line. Note that[3]

$$\frac{Dg_\alpha}{Dt} = \frac{\partial}{\partial t}\left(\frac{\partial x}{\partial \xi^\alpha}\right)\bigg|_{\xi^\beta} = \frac{\partial}{\partial \xi^\alpha}\left(\left.\frac{\partial x}{\partial t}\right|_{\xi^\beta}\right) = \frac{\partial v}{\partial \xi^\alpha} \tag{6}$$

Next, recall that any vector w may be expressed as

$$w = w^\alpha g_\alpha \tag{7}$$

where the w^α are known as contravariant components of the vector w (Ref. 2). In order to emphasize that the ξ^α are material curvilinear coordinates, the w^α are referred to as material contravariant components.

Vorticity Dynamic in Material Contravariant Components

Beltrami's vorticity equation (Ref. 1, Eq. 17.2) is given by

$$\frac{D}{Dt}\frac{\omega}{\rho} - \frac{\omega}{\rho}\cdot \text{grad} v = \frac{1}{\rho}\,\text{curl} a \tag{8}$$

where $\omega = \text{curl}\, v$ is the vorticity, ρ the density, and a the acceleration. Using Cauchy's equation of motion (Ref. 1, Eq. 6.7) along with Gibb's thermodynamics (Ref. 1, pp. 172-173) one obtains[4]

$$\frac{D}{Dt}\frac{\omega}{\rho} - \frac{\omega}{\rho}\cdot \text{grad} v = \frac{1}{\rho}\text{grad}\theta \times \text{grad} S + \frac{1}{\rho}\text{curl}\left(\frac{1}{\rho}\text{div} T_v\right) \tag{9}$$

where θ is the temperature, S the entropy, and T_v the viscous stress tensor. Equation (9) is a generalization to viscous flows of the vorticity equation of Vazsonyi (Ref. 1, Eq. 40.5). Inviscid nonconductive flows that are initially isentropic are inviscid and isentropic at all times[1] and Eq. (9) reduces to

$$\frac{D}{Dt}\frac{\omega}{\rho} - \frac{\omega}{\rho}\cdot \text{grad} v = 0 \tag{10}$$

or, in terms of contravariant components

$$\frac{D}{Dt}\left(\frac{\omega^\alpha}{\rho}g_\alpha\right) - \frac{\omega^\alpha}{\rho}\frac{\partial v}{\partial \xi^\alpha} = 0 \tag{11}$$

Using Eq. (6), one obtains

$$\frac{D}{Dt}\frac{\omega^\alpha}{\rho} = 0 \tag{12}$$

which shows that, for inviscid isentropic flows, the material contravariant components of the vorticity divided by the density are constant following a material point. Hence, in general, the vorticity is given by

$$\omega(\xi^\alpha,t) = \frac{\rho(\xi^\alpha,t)}{\rho(\xi^\alpha,t_0)}\omega^\beta(\xi^\alpha,t_0)g_\beta(\xi^\alpha,t) \tag{13}$$

or, recalling that $\rho J = \text{const}$ (Ref. 1, p. 133), where $J = \text{Det}(\partial x^i/\partial \xi^\alpha)$ is the Jacobian of the transformation given by Eq. (1),

$$\omega(\xi^\alpha,t) = \frac{J(\xi^\alpha,t_0)}{J(\xi^\alpha,t)}\omega^\beta(\xi^\alpha,t_0)g_\beta(\xi^\alpha,t) \tag{14}$$

If ξ^α concide with the Cartesian coordinates at $t=0$, Eq. (13) may be rewritten as

$$\omega = \frac{\rho}{\rho_0}\omega_0 \cdot \text{Grad} x \tag{15}$$

a result obtained by Cauchy in 1815 (Ref. 1, p. 152, Eq. 17.5). Equation (12), introduced in Ref. 3, has some advantages over Eq. (15) which are discussed herein.

In particular, it may be worth noting that for more general (viscous and/or nonisentropic) flows one obtains, following the same procedure,

$$\frac{D}{Dt}\left(\frac{\omega^\alpha}{\rho}\right) = z^\alpha \tag{16}$$

where z^α are the contravariant components of $(1/\rho)$ curl a [see Eqs. (8) and (9)]. To this author's knowledge, Eq. (16) is a new result.

Vortex Theorems for Inviscid Isentropic Flows

Some classical vortex theorems may be obtained (more directly than in the classical approach[1]) as an immediate consequence of the above results. The first is that, for inviscid isentropic flows, a vortex line is a material line. In order to prove this, the coordinates ξ^α are chosen such that the ξ^1 lines (and, hence, the base vectors g_1) are parallel to the vorticity field at time $t=t_0$. This implies $\omega^2 = \omega^3 = 0$ at $t=t_0$ and, according to Eq. (12), $\omega^2=\omega^3=0$ at all times, or

$$\omega(\xi^\alpha,t) = \frac{J(\xi^\alpha,t_0)}{J(\xi^\alpha,t)}\omega^1(\xi^\alpha,t_0)g_1(\xi^\alpha,t) \tag{17}$$

Hence, ω is always parallel to the ξ^1 lines, which, as mentioned above, are by definition material lines; therefore, vortex lines are material lines (in agreement with Ref. 1, p. 152).

Another classical result easily obtained from Eq. (13) is related to vortex stretching. Using Eq. (17) one obtains

$$\frac{|\omega|/\rho}{|\omega_0|/\rho_0} = \kappa \tag{18}$$

where

$$\kappa = |g_1|/|g_1|_0 = |dx|/|dx|_0 \tag{19}$$

is the stretching factor of the vortex line at the given material point.

It may be worth noting that if we introduce a small material cylinder (around an element $d\boldsymbol{x}$ of a vortex line) having a cross section dA and a constant mass

$$dm = \rho dA\ |d\boldsymbol{x}| = \rho_0 dA_0\ |d\boldsymbol{x}|_0 \tag{20}$$

one obtains, using Eqs. (19) and (20),

$$|\omega| dA = |\omega_0| dA_0 \tag{21}$$

in agreement with Kelvin's theorem, which states that $d\Gamma/dt = 0$, with

$$\Gamma = \oint_C \boldsymbol{v}\cdot d\boldsymbol{x} = \iint_\sigma \boldsymbol{\omega}\cdot\boldsymbol{n}d\sigma \tag{22}$$

where C is a material contour and σ any surface with C as its boundary.

Computational Applications

The above results also have important computational applications. Consider for simplicity the case of an inviscid and isentropic flow (e.g., as indicated above, the case in which viscosity and conductivity are negligible and the flowfield is isentropic at time $t=0$). In this case the vorticity is given by Eq. (13) at all times. Assume that we know the vorticity field at time $t=t_0$ and the functions $\boldsymbol{x}(\xi^\alpha,t)$ at times t_0 and t_1. Whereas these pieces of information are not sufficient to evaluate the velocity field, nonetheless they are sufficient to evaluate the vorticity field at time t_1. Indeed, from $\boldsymbol{x}(\xi^\alpha,t_0)$ we may evaluate the vectors $\boldsymbol{g}_\beta$ at $t=t_0$ and hence the contravariant components $\omega^\beta(\xi^\alpha,t_0)$ as well as $J(\xi^\alpha,t_0)$. On the other hand, the knowledge of $\boldsymbol{x}(\xi^\alpha,t_1)$ (which may be obtained with the procedure outlined below), allows for the evaluation of $\boldsymbol{g}_\beta$ and J at $t=t_1$ and hence of the vorticity field, via Eq. (14).

Therefore, Eq. (14), besides its theoretical applications, may be exploited for computational purposes. For instance, consider the particularly complex problem of wake dynamics for a helicopter rotor is an incompressible flow. For this problem the vorticity is concentrated in the boundary layer and the wake. Within the wake the flow may be treated as inviscid.[5] Once the locations of material grid points are evaluated at a given time step, the direction and intensity of the vorticity field can be evaluated immediately using Eq. (14) [or the simpler Eq. (17) if the points of the grid are initially aligned with the vortical lines]. Once the vorticity is known, the velocity is evaluated using the Biot-Savart law (with the addition of the gradient of a scalar potential to satisfy normal boundary conditions; see Ref. 6 for details). This provides an extension to distributed vorticity of the formulation introduced in Ref. 5 for potential flows. In general, if the assumption of inviscid isentropic flow is not adequate, ω^α may be evaluated, step by step, using Eq. (16).

Conclusions

In summary, Cauchy's classical result given by Eq. (15) has been reinterpreted in terms of contravariant components in a material curvilinear coordinate system. Also, Vaszonyi's vorticity equation was extended to viscous flows and Cauchy's result was extended to such flows [Eq. (16)]. The elimination of the unnecessary constraint that the ξ^α coordinates coincide, at $t=0$, with the Cartesian components has facilitated the derivation of classical vortex theorems. The use of the results for the computational solution of the generalized vorticity equation was also outlined.

Acknowledgments

This work was supported by the Army Research Office (Contract No. DAAG26-83-K-0050 to Boston University). The author wishes to thank Dr. Guido Sandri and Dr. Norman D. Zabusky for their encouragement and recommendations on the writing of this Note.

References

[1]Serrin, J., "Mathematical Principles of Classical Fluid Mechanics," *Encyclopedia of Physics*, edited by S. Flugge, Vol. VIII/1, *Fluid Dynamics I*, coedited by C. Truesdell, Springer-Verlag, Berlin, 1959, pp. 125-263.

[2]Sokolnikoff, I.S., *Tensor Analysis*, 2nd Ed., John Wiley & Sons, New York, 1964.

[3]Morino, L., "A Finite-Element Method for Rotational Incompressible Aerodynamics," Boston University, College of Engineering, TN-74-04, Dec. 1974.

[4]Morino, L., "A General Scalar/Vector Potential Formulation for Compressible Viscous Unsteady Flows," ICARUS, Boston, TR-84-01, Aug. 1984.

[5]Morino, L., Kaprielian, Z. Jr., and Sipcic, S.R., "Free Wake Analysis of Helicopter Rotors," *Vertica*, Vol. 9, June/July 1985.

[6]Morino, L. and Bharadvaj, B., "Two Methods for the Free Wake Analysis of Helicopter Rotors," Boston University Center for Computational and Applied Dynamics, TR-85-02, Aug. 1985.

Turbulent Boundary Layers with Vectored Mass Transfer

Gustave J. Hokenson*
The Hokenson Company, Los Angeles, California

THE excellent data presented in Ref. 1 provide substantial guidance in the understanding of many of the subtleties of turbulent boundary-layer (TBL) flows with blowing, and the modeling thereof. In an effort to peel away quantitatively the various interacting elements of the flow, it is proposed that the following approach may be useful for exploiting such data in model development. If the Reynolds-averaged equations of motion are further locally spatially averaged in horizontal planes, with a scale large relative to the porosity yet small relative to the boundary-layer growth, the discrete jet blowing may be converted into its distributed mass transfer equivalent. For example,

$$\bar{u} = \langle \bar{u} \rangle + \bar{u}^\circ \tag{1}$$

where $\bar{u}$ is the time-averaged streamwise velocity and $\langle\ \rangle$, $(\ )^\circ$ denote the local spatial average and the variation therefrom, respectively. As a result of this local spatial averaging, the governing equation nonlinearities generate extra (Reynolds stress-like) terms due to the spatial variations. (Note that a piecewise constant filter function is used such that no Leonard terms appear.) In addition, the average normal velocity at the surface is now simply related to the total flow rate and area, which reduces the local jet-like values of wall normal velocity by the open area ratio of the surface, while the average skin friction is the weighted sum of the solid surface plus open surface contributions. Finally, the doubly averaged streamwise velocity and turbulence at the wall are now nonzero.

It appears that the location in the boundary layer at which $\langle \bar{u}^\circ \bar{v}^\circ \rangle = 0$ (where $\bar{v}$ is the time-averaged normal velocity) defines a level that divides the relatively weakly sheared

Received June 3, 1985.

*Chief Scientist. Member AIAA.